Combining risk measures to overcome their limitations - spectrum representation of the sub-additivity issue, distortion requirement and added-value of the Spatial VaR solution: An application to Regulatory Requirement for Financial Institutions

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Combining risk measures to overcome their limitations - spectrum representation of the sub-additivity issue, distortion requirement and added-value of the Spatial VaR solution: An application to Regulatory Requirement for Financial Institutions

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Abstract

To measure the major risks experienced by financial institutions, for instance Market, Credit and Operational, regarding the risk measures, the distributions used to model them and the level of confidence, the regulation either offers a limited choice or demands the implementation of a particular approach. In this paper, we highlight and illustrate the paradoxes and issues observed when implementing an approach over another, the inconsistencies between the methodologies suggested and the problems related to their interpretation. Starting from the notion of coherence, we discuss their properties, we propose alternative solutions, new risk measures like spectrum and spatial approaches, and we provide practitioners and supervisor with some recommendations to assess, manage and control risks in a financial institution[3].

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1 Introduction

The ECB-SSM[4], the EBA [5] and the Basel Committee have been constantly reviewing the methodological framework of risk modelling for the past 20 years. In this paper, we analyse some of the issues observed when measuring the risks that would be worth addressing in future regulatory documents.

1.1 Problematic

During the 2007/08' crisis[6], the failure of models and the lack of capture of extreme exposures have led regulators to change the way risks were measured, either by requiring financial institutions to use particular families of distributions (Gaussian (BCBS (2005)), sub-exponential (EBA (2014*b*))), or by changing the way dependencies are captured (EBA (2014*b*)) or by suggesting a shift from the Value-at-Risk (VaR)[7] to sub-additive risk measures like the Expected Shortfall (ES)[8] (BCBS (2013)). Indeed, inappropriate risk modelling had played a major role during the crisis which began in 2008 either as a catalyst or trigger. The latest changes proposed by the authorities have been motivated by the will to come closer to the reality of financial markets.

[4]European Central Bank - Single Supervisor Mechanism

[5]European Banking Authority

[6]Known as the subprime crisis.

[7]Given a confidence level $p \in [0,1]$, the VaR associated to a random variable X is given by the smallest number x such that the probability that X exceeds x is not larger than $(1-p)$

$$VaR_p = \inf(x \in \mathbb{R} : P(X > x) \leq (1-p)). \tag{1.1}$$

[8]For a given p in $[0,1]$, η the VaR_p, and X a random variable which represents losses during a pre-specified period (such as a day, a week, or some other chosen time period) then,

$$ES_p = E(X|X > \eta). \tag{1.2}$$

In a recent paper we have discussed the importance of the choice of the distributions in measuring the risks (Guégan and Hassani (2016)). In this paper, in a first step, we go beyond the findings of our previous paper, and discuss the concept of coherence[9] and in particular the impact of the notion of sub-additivity, which interested many researchers in the past 20 years (Artzner et al. (1999), Dhaene et al. (2008) and Jorion (2006)), complying with regulatory requirements (BCBS (2011a), BCBS (2011b), BCBS (2013), EBA (2014a)). Then in a second step, we discuss the spectral approach, the spectrum representation of a risk measure (i.e. the value of the risk measure for each and every percentile of a given distribution), the distribution of risk measures themselves, and the combination of risk measures to create ranges allowing to capture the "true" risk instead of an approximation of this one by a point.

Thus, the purpose of this paper is to address the methodological aspects of the regulatory framework related to risk modelling and its evolution since 1995, focusing on supervisors' strong incentive to use: (i) specific distributions to characterise the risks, (ii) specific risk measures, (iii) specific associated confidence level, and to apply these strategies independently from each other. We argue that approaches proposed by the regulator engender a bias (positive or negative) in the assessment of the risks, and consequently a distortion in both the corresponding capital requirements and the management decision taken since the problem of the measurement is not dealt with in its entirety, and as such we question the motivation of the regulator.

Some of the following points are addressed in this paper: (i) Is the choice of a particular risk measure ensuring conservativeness? (ii) When moving from a VaR_p to sub-additive risk measures such as the ES_p, for which distributions is the sub-additivity[10] property fulfilled given that we consider several risk factors? (iii) Given that each risk type is modelled based on different

[9]This concept is a particular case of the notion of convex measure that we do not address in this paper, Follmer and Schield, 2002)

[10]A coherent risk measure is a function $\rho : \mathcal{L}^\infty \to \mathbb{R}$:

- Monotonicity:If $X_1, X_2 \in \mathcal{L}$ and $X_1 \leq X_2$ then $\rho(X_1) \leq \rho(X_2)$
- Sub-additivity: If $X_1, X_2 \in \mathcal{L}$ then $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$
- Positive homogeneity: If $\lambda \geq 0$ and $X \in \mathcal{L}$ then $\rho(\lambda X) = \lambda \rho(X)$
- Translation invariance: $\forall k \in \mathbb{R}$, $\rho(X + k) = \rho(X) - k$

distributions and using different p-s, how can the sub-additivity criterion be fulfilled? Is that really important in practice? (iv) Should we combine risk measures? (v) Should we focus on ranges of risk values rather than unique value? These different points are linked to the choice of a particular distribution, to the choice of the confidence level p and to the risk measure itself.

The regulatory documents state with respect to market risk - since 1995 (BCBS for instance) - that "the VaR risk measure is inadequate for measuring the risks because it does not take into account the extreme events" and also "one of the problems of recognising banks"' value-at-risk measures as an appropriate capital charge is that the assessments are based on historical data and that, even under a 99% confidence interval, extreme market conditions are excluded"[11]. To confirm this fact, in the Consultative Document concerning the Fundamental review of the trading book (BCBS (2013)), the Basel Committee proposes "to move from Value-at-Risk (VaR) to Expected Shortfall (ES) as a number of weaknesses have been identified using VaR for determining regulatory capital requirements, including its inability to capture tail risk". The Committee has agreed "to use a 97.5th ES for the internal models-based approach and to use it to calibrate capital requirements under the revised market risk standardised approach". We may argue that this modification has been decided with the Gaussian distribution in mind as the values of the 97.5th ES and the 99th VaR are very close to each other.

In these documents the regulator states that the choice of the VaR as a risk measure does not take into account extreme values. This statement is not correct as the choice of the VaR is not the issue; it is the choice of the underlying distribution with which the associated quantile is evaluated that determines if the extreme events are captured or not. This point actually implies a second question about what an extreme event is and answering it would suppose a complete information set. Then in 2013, it seems that the regulator thought that the use of the ES instead of the VaR would be more effective to capture the most relevant information to measure the risks. This is not necessarily true as once again, it depends on the choice of the distributions used for the computation of this ES. Nevertheless, we know that this last measure is more interesting than the VaR when considering the same distribution because it provides better information concerning the amplitude of the risk, but if the fitted distribution is inappro-

[11] There is no mean to know for sure that the most extreme market conditions have been met.

priate[12] the problem of capturing extreme events remains the same. Besides, the choice of the level of confidence, for instance 97.5 is also arbitrary (this point will be illustrated in the next section). Indeed, why did the regulator move from 99% (in 1995) to 97.5 % (in 2013)? - Why did they not suggest 95% or another value p?

Another point is considered by regulators for modelling operational risk (EBA (2014*b*)[13]). Indeed, they consider that a "risk measure means a single statistic extracted from the aggregated loss distribution at the desired confidence level, such as Value-at-Risk (VaR), or shortfall measures (e.g. Expected Shortfall, Median Shortfall)". This definition is particularly, reductive, limiting and dangerous. How can the risk measures computed for different factors with different levels be aggregated? If we use the ES measure, it loses its sub-additivity property in that latter case. Thus, other approaches could be more robust and realistic, for instance the use of spectral measure, or a spectrum of the previous risk measures.

It appears that some documents are too prescriptive, preventing banks from going beyond the proposals and focusing more on the capital calculations than on the risk measurement and management itself. Regarding the calculation of the capital requirement from the knowledge of the risk factors, the main points concern the choice of the distribution, the choice of the risk measure and the choice of p: these choices are not studied in a uniform manner and the approach proposed by regulators does not constitute a robust approach for measuring the risks of financial institutions.

In Section Two, we investigate the notion of sub-additivity for a risk measure showing that this property do not only depends on the choice of the risk measure but also depends on the choice of the distribution. We illustrate the point that the restriction imposed by regulators prevents a reliable approach to measure the risk. We illustrate our statements with examples. In Section Three we show that it is the choice of the distributions which is definitively the key point in risk modelling. Then in Section Four, we propose an innovative way to look at risk measures,

[12]Goodness-of-fit test are only relevant with respect to the information considered. Besides, though a distribution might be appropriate given a data set, if the underlying information evolves (in other words if the sample changes), the distribution might not be valid anymore according to that test.

[13]The discussed philosophy is also implied in the final version of the document.

working in several directions; discussing the spectral measure, introducing a new one, the spatial risk measure which relies on a confidence region, or using a measure based on the distortion of the distribution in order to capture the multi-modal aspect of some risk factors. Section Five concludes.

2 Sub-additivity property: a real added-value for risk management?

The concept of sub-additivity which has been largely studied in the past 20 years appears interesting if we consider that the measure of the risk of a portfolio[14] is obtained when we calculate the risks of each factor of this portfolio. This is a very restrictive approach for measuring these exposures. An appropriate solution would be to use a multivariate quantile approach based on copula or vines (Guégan and Maugis (2010a), Guégan and Hassani (2013), etc). Nevertheless, if we maintain this method of assessing the risks, the idea of the sub additive risk measure is that the risk measure of the sum has to be smaller than the sum of the risk measures obtained for each factor taken individually, and following the works of Artzner et al. (1999), it seems that this property is only verified by the ES risk measure in all cases. In fact, this property is also verified by the VaR measure, only now it depends on the distribution used (Degen and Embrechts (2008)). Indeed, the VaR is known to be sub-additive (i) for stable distribution, (ii) for all log-concave distributions, (iii) for the infinite variance stable distributions with finite mean, (iv) and for distribution with Generalised Pareto Distribution type tails when the variance is finite (i.e. when the shape (usually denoted ξ) is inferior to 0.5). The non sub-additivity of the VaR can occur (i) when P&L distributions are greatly skewed; (ii) when the dependency between factors is highly asymmetric, and (iii) when underlying risk factors are independent but highly heavy-tailed. To illustrate our purpose, we have selected a data set provided by a European bank representing "Execution, Delivery and Process Management" risks from 2009 to 2014. "Execution, Delivery and Process Management" risk is a sub-category of operational risk[15]. This data set is characterised by a right skewed (positive skewness) and leptokurtic distribution. Note that Operational Risk losses are defined on $]-\infty, 0[$, as a result, in the following com-

[14] The term portfolio is here used *stricto sensus* and not necessarily as a combination of assets.

[15] In our demonstration, the data set which has been sanitised here is not of particular importance since the same data set has been used for each and every distribution tested.

putations and discussions, the sign will not be considered and the data treated as positive values.

In order to follow the regulatory guidelines, we choose to fit on this data set some of the distributions prescribed and also others which seem more appropriate considering the properties of the data set. We retained eight distributions. They are estimated (i) on the whole sample: the empirical distribution, the lognormal distribution (asymmetric and medium tailed), the Weibull distribution (asymmetric and thin tailed), a Generalised Hyperbolic (GH) distribution (symmetric or asymmetric, fat tailed on an infinite support), an α-Stable distribution (symmetric, fat tailed on an infinite support), a Generalised Extreme value (GEV) distribution (asymmetric and fat tail), (ii) on an adequate subset: the Generalised Pareto (GPD) distribution (asymmetric, fat tailed) calibrated on a set built over a threshold, a Generalised Extreme Value (GEVbm) distribution (asymmetric and fat tailed) fitted using maxima coming from the original set. The whole data set contains 98082 data points, the sub-sample used to fit the GPD contains 2943 data points and the sub-sample used to fit the GEV using the block maxima approach contains 3924 data points. The objective of these choices is to evaluate the impact of the selected distributions on the risk representation, i.e. how the initial empirical exposures are captured and transformed by the model. It is interesting to note that using empirical distributions instead of fitted analytical distributions could be of interest as the former one captures multi-modality by construction. Unfortunately, this solution was initially rejected by regulators who consider this non-parametric approach not able to capture tails properly: looking at table 11 this idea might be a false statement. However, recently the American supervisor seems to be re-introducing empirical strategies in practice for CCAR[16] purposes.

Table 1 exhibits parameter estimates for each distribution selected[17]. The parameters are estimated by maximum likelihood, except for the GPD which implied a POT (Guégan et al. (2011)) approach and the GEV fitted on the maxima of the data set (maxima obtained using a block maxima method (Fisher and Tippett (1928), Gnedenko (1943))). The quality of the adjustment is measured using both the Kolmogorov-Smirnov and the Anderson-Darling tests. The results presented in Table 1 show that none of the distributions is adequate. This is usually the case

[16] Comprehensive Capital Analysis and Review

[17] In order not to overload the table the standard deviation of the parameters are not exhibited but are available upon request.

when fitting uni-modal distributions onto a multi-modal data set. Indeed, the multi-modality of distributions is a frequent issue when modelling risks, in particular operational risks, as the units of measures usually combine multiple kinds of incidents; for instance a category combining external frauds will contain the fraud card on the body, commercial paper fraud in the middle, cyber attack and Ponzi scheme in the tail, but we have also observed a similar pattern using market data. [18]

In the introduction we indicated that the regulator recommends (for market risk in particular) the use of the ES instead of the VaR because the former is sub-additive, property unverified by the VaR. In the following, we question these assertions showing that, even if it is true that the ES is sub-additive, (i) the VaR also has this property for a lot of distributions as we have mentioned before; (ii) the sub-additivity property can be verified for some values of p, and not verified for others; (iii) the sub additivity of the VaR is very often verified for fat-tailed distributions (note that the problem is addressed here in a context different from Daníelsson et al. (2013); (iv) the sub-additivity is not verified anymore for the ES when we aggregate them. We illustrate these different facts making some simulations computing $\mathrm{VaR}_p(X+Y)$ and $\mathrm{VaR}_p(X) + \mathrm{VaR}_p(Y)$ for X and Y two risk factors. We proceed in the following way (we will proceed similarly for the ES):

- As $\mathrm{VaR}_p(X)$ is a quantile, $p \in [0,1]$, the entire spectrum of the VaR has been built, considering the inverse of the cumulative distribution function, sometimes referred to as the quantile function. Summing $\mathrm{VaR}_p(X)$ and $\mathrm{VaR}_p(Y)$ for each value of p provides us with $\mathrm{VaR}_p(X) + \mathrm{VaR}_p(Y)$.

- To obtain $\mathrm{VaR}_p(X+Y)$, another approach is adopted. In a first step we randomly generate X and Y using specific distributions. Then X and Y are aggregated. The resulting cumulative distribution function is built and its inverse provides the spectrum of $\mathrm{VaR}_p(X+Y)$[19].

In Table 2 we provide the values obtained for both the VaR and the ES for fully correlated random variables. It is interesting to note that the risk measures obtained on fully correlated random variables and the sum of the risk measures obtained univariately are really similar. This

[18] It could be more appropriate to consider empirical distributions than fitted analytical distributions as it may help to capture multi-modality. We will come back to this last point in Section 4.

[19] We acknowledge the numerical error that this process may engender, though this one appears negligible here.

means that as soon as we sum the VaR obtained on two variables we mechanically assume an upper tailed correlation of the random variables. Therefore, as well as being conservative, the sum of univariate VaRs taken at the same level prevents the capture of any diversification benefit. Fully correlated random variables do not embed any diversification benefit by definition. Consequently, are we really comparing what is comparable i.e. mechanically highly correlated factors with potentially independent or negatively correlated ones? One may argue that the sub-additivity inequality compares results obtained from a multivariate distribution on one side with highly correlated variables on the other, and this may bias our understanding of the phenomenon in practice.

Then we work with the data sets we have previously introduced. We randomly generated values from the distribution fitted before and combined them two by two. By carrying out this process we generated some random correlations and incidentally some diversification. Then, we compared the risk measures obtained from the combination of random variables and the sum of the risk measures computed on the random variables taken independently. Tables 3, 4, 5 and 6 are exhibiting the differences between $\mathrm{VaR}(X) + \mathrm{VaR}(Y)$ and $\mathrm{VaR}(X + Y)$ where X and Y are random variables successively following each of the distributions presented earlier. These tables are now analysed.

From Table 3, for fixed p, we observe that the VaR is never sub-additive if the lognormal distribution is associated with a GPD; while if the lognormal distribution is associated with any of the others, the VaR is usually sub-additive in the tails but not at the end of the body part. Note that if the lognormal is associated with an identical lognormal, the differences we have observed are only due to numerical errors related to sampling. We expect the two values to be absolutely identical. However, it is interesting to note that the random generation of numbers can be at the origin of non sub-additive results. An identical analysis can be done on other combinations such as those containing the Weibull distribution as observed in Table 4. From Table 5 it appears that when the GPD has a positive location parameter, this prevents any combination from being sub-additive, because by construction the 0th percentile of the GPD is equal to the location parameter which should, according to Pickand's theorem (Pickands (1975)), be sufficiently high. At the 95th percentile, the VaR is always sub-additive whenever a lognormal distribution is involved, except if it is combined with a GPD. For the other distributions, it is not always true. For example, the VaR obtained after combining a Weibull and a GEV fitted on the whole sample is not sub-additive. Table 6 shows that the use of an Alpha-Stable combined with any other distribution, except for the GPD, provides sub-additive risk measures at the 99% level. Other examples are provided in Table 6 with the Weibull distribution.

Building the ES always respects the sub-additivity property (see Tables 7, 8, 9 and 10), contrary to the VaR for which this property is not always verified and depends on the underlying distribution as discussed previously: the results for the ES can be compared to those obtained with the VaR looking at Tables 3 and 7, Tables 4 and 8, then Tables 5 and 9 and finally Tables 6 and 10. It is interesting to note that if we combine two ES measures taken at two different levels of confidence p, the ES may not be sub-additive anymore. This is a point that the regulators do not discuss when they ask to aggregate the results provided by each risk measure. We fail to understand the consistency of their mindful process. What is the point of demanding a coherent risk measure when this property fails when aggregating the risks measured as required by the regulation (see first section)? This issue is particularly important for risk managers, since the level of confidence prescribed in the regulation guidelines is different from one risk factor to another and seems totally arbitrary though some rational may support them.

In parallel, Figures 1 to 5 allow a more discriminating analysis of the behaviour of the component $VaR_p(X+Y)$ versus $VaR_p(X)+VaR_p(Y)$. In Figure 1, we show that the sub-additivity property is only verified for high percentiles when we combine a Weibull and a GH distribution, i.e. for $p > 90\%$. In addition, the gap tends to widen as the percentiles increase. Figure 2 exhibits a non sub-additive VaR from the 95th percentile, when we use the combination of an α-Stable distribution and a GEV fitted with the block maxima method, but the differences are not as great as in Figure 1. Figure 3 shows that combining two identical distributions does not always produce sub-additive risk measures though it should always be the case: this can be due to numerical errors caused by the random generation of data points and the discretisation of the distribution. In Figures 4 and 5 we observe that the VaRs obtained from the combination of an Alpha-Stable distribution and a GH distribution or an Alpha-stable distribution and a GEV distribution calibrated on maxima are never sub-additive below 70%. For comparison purposes, Figures 6 and 7 illustrate the fact that the combination of two elliptical distributions (respectively the Gaussian and the Student-t distributions) always leads to sub-additive VaRs (given p).

3 Role of the distributions in the computation of VaR and ES measures

In the previous Section analysing the role of the sub-additivity concept, we point out the importance of the choice of the distributions. In this section we illustrate this influence on the univariate risk measure computation. Table 11 provides the values obtained for the VaR_p and the ES_p computed from the eight distributions either fitted on the whole data set or some subsamples. We also illustrate the fact that an *a priori* on the choice of a distribution provides arbitrary results which can be disconnected from reality, and also the importance to use an appropriate approach for estimation.

From Table 11 we note that the values provided by VaR_p can be larger than the values derived from an ES_p and conversely. We observe that the results obtained from the GPD and the α-stable distributions are of the same order. Second, the differences between the GPD and the GEV fitted on the block maxima are huge, illustrating the fact that, despite being two extreme value distributions, the information captured is quite different. The ES calculations are also linked to the distribution used to model the underlying risks. Looking at Table 11, at 95%, we observe that the ES goes from 1979 for the Weibull to 224 872 for the GPD. Therefore, depending on the distribution used to model the same risk, at the same p level, the ES obtained is completely different. The corollary of that issue is that the ES obtained for a given distribution at a lower percentile will be higher than the ES computed with another distribution at a higher percentile. For example, Table 11 shows that the 90% ES obtained from an Alpha-Stable distribution is much higher than the 99.9% ES computed on a lognormal distribution.

Consequently, one question arises. What should the regulator ask to use: the VaR or the Expected Shortfall? To answer this question, we can consider several points: (i) Conservativeness: Regarding that point, the choice of the risk measure is only relevant for a given distribution, i.e.

for any given distribution the VaR_p will always be inferior to the ES_p (assuming only positive values) for a given p. But, if we consider two distributions to characterise a risk it may happen that for a given level p, the VaR_p obtained from a distribution is superior to the ES_p for another distribution. For example, Table 11 shows that the 99.9% VaR obtained using the GH distribution is superior to the ES obtained for the Weibull or the lognormal distributions at the same level p; (ii) Distribution and confidence level p impacts: Table 11 shows that potentially a 90% level ES obtained from a given distribution is larger than a 99.9% VaR obtained with another distribution, e.g. the ES obtained from a GH distribution at 90% is higher than the VaR obtained from a lognormal distribution at 97.5%. Thus is it always pertinent to use a high value for p? (iii) Parameterisation and estimation: the impact of the calibration of the estimates of the parameters is not negligible (Guégan et al. (2011)), for instance when we fit a GPD. Indeed, in that latter case, due to the instability of the estimates of the threshold, the practitioners can largely over-represent the risks. The fitting, for instance, of a Weibull distribution using a whole sample in place of a set of maxima (build for example using block maxima method) is totally counterproductive (if we want to model correctly the tails) because the values obtained for the VaR with this distribution are difficult to interpret.

4 Extending traditional approaches: Spectral measure, Spectrum, Spatial risk measure and Distortion

4.1 Spectral Risk Measure vs Spectrum

Discussing the importance of the choice of the distributions and also the choice of the level p in the previous sections we point the fact that it is nearly impossible to decide the appropriate level p to perfectly capture the risks associated to any factor, suggesting the idea that these two notions are intrinsically linked. Indeed in Tables 2 and 11, we exhibit for different p the values of both VaR and ES for various distributions and it appears that using multiple levels could be informative in terms of real exposures. This is the objective of two other concepts: the spectral risk measures and the spectrum associated to each risk measure. We specify these two notions and show how these can be built.

A spectral risk measure is obtained considering a weighted average of outcomes. By construction, it is a coherent risk measure. Spectral measures found their usefulness in the fact that they can be related to risk aversion through the weights chosen to reflect the possible risk exposures. A formal representation of spectral risk measure ρ (Acerbi, 2002) is a function defined from $\mathcal{L} \to \mathbb{R}$ such that:

$$\rho(X) = -\int_0^1 \phi(p)F^{-1}(x)(p)dp \tag{4.1}$$

where ϕ is positive or null, non-increasing, right-continuous, integrable function defined on $[0,1]$ such that $\int_0^1 \phi(p)dp = 1$ and $F(x)$ is the cumulative distribution function for x. Any spectral risk measure satisfies the conditions for coherence making them useful in practice as well as law-invariance and comonotonic additivity[20] (Adam et al. (2007)).

The ES is a spectral risk measure for which $\phi(p) = 1$, $\forall p$ in (4.1). The limitation of the spectral ES is that all the weights are uniform whatever p by definition, however, this insures its coherence. When financial institutions provide the ES, i.e., a unique value to the regulator to represent its risks, it provides a spectral risk measure. Consequently, up to certain extent, the aggregation of the risks computed for each department within a financial institution can be considered as a spectral risk measure. Unfortunately, as introduced, the regulation requires these risks to be assessed for different p levels, set by the regulator: 99,9% for operational risk, 95% for market risks, etc. In that case the property of coherence of the risk measure cannot be ensured anymore, however the coherence of the risk measure might still be achieved, as presented in Guégan et al. (2016), depending on the distributions selected to represent the risks.

Note that as a quantile, and not a combination of them, the VaR is not a spectral risk measure but may have a spectral representation. The general representation of spectral risk measures is more flexible than the particular case of the ES. Indeed, for a given risk factor, we can associate different weights to different levels depending on if we want to privilege the body or the tail of the distribution. This can be equivalent to fitting a multi-modal distribution to the data characterising the risk factor, and then to recalculate the risk measure (see below the section on the distortion). In Tables 2 and 11, we have shown the great variability of the values obtained for

[20] Law-Invariance: for any combination of risks X and Y with respective cumulative distribution functions $F(x)$ and $F(y)$, if $F(x) = F(y)$ then $\rho(X) = \rho(Y)$; comonotonic additivity: for every comonotonic random variables X and Y, $\rho(X+Y) = \rho(X) + \rho(Y)$.

the VaR for different p. The spectral approach is supposed to provide a more complete measure of risk as soon as we choose to associate to each quantile a weight. In fine, we get an unique value which is different of the concept of the spectrum that we are going to describe now.

We call spectrum of a risk measure, the set of all the values we get for this risk measure given a set of levels p_i. We exhibit in Figure 8 an illustration of a spectrum of the ES for a given risk factor for different distributions and for $p_i \in [0.70, 0.999]$. This graph provides several information: the role of the distribution, the influence of the choice of p, the large difference depends on if we are close to the body of the distributions or in the tails (Note that the differences can be extremely significant depending on the distribution chosen).

Indeed, while the use of several levels $p_i, i = 1, \cdots, k$ allows a spectrum representation of the risk measures (VaR or ES) and could be more interesting for risk management than the approach proposed by regulators using the idea of spectral measure which combines distribution and confidence level. Indeed, the 70% ES of some combinations may lead to a much higher value than the 99.9th (Table 8, WE-GPD vs WE-GH); on the contrary we provide in Tables 12, 13 and 14, the differences $VaR(X) + VaR(Y) - VaR(X + Y)$ for several distributions. In Table 12 we use Weibull and a lognormal distributions, in Table 13 Weibull and α-stable distributions and in Table 14 two GEV distributions. We do this exercise for $90\% < p < 99.9\%$ in Tables 12 and 13, and for $1\% < p < 99.9\%$ with a step of 1% in Table 14. In that last table when the values are positive, the VaR is sub-additive, when the values are negative it is not sub-additive. The turning points are highlighted in bold. This provides an interesting picture of the property of these distributions. The spectrum of the VaR given in these tables provides good information in terms of risk management; indeed, it shows that relying directly on risk measures to evaluate a capital requirement may not be representative of the risk profile of the target entity. In fact, it can even be misleading, as from one p_i to another because the risk measures may have dramatically different orders. The spectrum of the VaR approach shows a risk measure obtained at a particular level cannot be representative of the whole risk profile, and assuming the contrary could lead to dreadful failures and inappropriate management decision. Thus we can incite risk managers to construct the spectrum to have a better understanding of these risks.

4.2 A Spatial Risk Measure: the SVaR

In this section, following an idea developed in a paper of Guégan et al. (2016), we present a way of measuring risks spatially, capturing the intrinsic uncertainty of the VaR both vertically and horizontally. By vertically, we consider the confidence interval (CI) of the VaR at a given percentile and horizontally by assuming that the level p is not properly evaluated (usually, supposed to represent the 95th, the 97.5th or the 99.9th - see section 1 for the regulatory requirements). The construction of the CI is specifically described in the following paragraphs. Therefore, instead of using a value for p, a range, in which p lies, is selected. For example for the 95th, we may want to select a range from the 92nd to the 98th. The combination of the vertical CI and the horizontal one leads to a quadrilateral figure giving the area in which the true VaR is supposed to lie.

This area - delimited by VaR_{p_i} and the upper bound of CI_{q_i} - corresponds to a Spatial risk measure we propose to use as an alert indicator. Indeed, having the VaR for different p provides us with the spectrum of the VaR. Then using a set of confidence intervals around this spectrum provides us with an acceptable range of variation for the VaR_p-s. Only considering the upper bounds on the confidence intervals gives this spatial VaR. The upper envelop (or the lower depending on the sign) allows us to use multiple values at a particular percentile level. To build the confidence interval we use a recent result introduced by Guégan et al. (2016).

In their paper, Guégan et al. (2016) studied the asymptotic Gaussian property of the distribution of the estimator $\widehat{VaR_p}$ of VaR_p, which allows to build a confidence interval CI_q around VaR_p. Since the convergence speed of this estimate depends on the underlying unknown distribution f, the sample size n of the data set and the confidence level p of VaR_p, the paper provides a comprehensive analysis of application for finite samples with a panel of distributions on risk data. The confidence interval they propose is the following:

$$\left[VaR_p - Z_{1-\frac{q}{2}}, \quad VaR_p - Z_{\frac{q}{2}}\right] \tag{4.2}$$

where $Z_q = \Psi_q^{-1}(\sqrt{n}\widehat{\omega}^\sharp)$. Ψ and $\widehat{\omega}^\sharp$ are provided in expressions below.

$$\Psi(\sqrt{n}\omega^{\sharp}) = 1 - \Phi(\sqrt{n}\omega^{\sharp}) \qquad \omega^{\sharp} = \omega + \frac{1}{n\omega}\log\frac{1}{\psi(-\omega)} \tag{4.3}$$

$$\psi(-\omega) = \frac{\omega(t-1)}{t-p}\left(\frac{r_0}{1-p}\right)^{\frac{1}{2}} \qquad \omega = -\sqrt{2h(t)}sign(t-p) \tag{4.4}$$

$$h(t) = p\log\frac{p}{t} + (1-p)\log\frac{1-p}{t} \tag{4.5}$$

where $sign(x) = 1$ if $x \geq 0$ and $sign(x) = -1$ otherwise. Besides, ϕ represent the Gaussian distribution.

In their article Guégan et al. (2016) call this approach a saddle point approximation for the confidence interval. It is based on a result of Zhu and Zhou (2009) which provides an asymptotic convergence of the estimate of the VaR_p towards its true value with a relatively fast rate of convergence, which is maintained in case of finite samples.[21].

In order to show the power of this new spatial risk measure, we illustrate its use on an example. Here, we are interested in the lower envelop as considering a distribution defined on an infinite support $(-\infty, +\infty)$, the losses are represented by negative values. Figure 9 exhibits the construction of the Spatial VaR using S&P 500 data from 01/01/2008 to 31/12/2008. The abscissa provides the "p"-s at which the VaR estimate (in ordinate) has been calculated. On the left, we present an truncated axis presenting the "q"-s. Here the Spatial VaR tells us in which range the 97^{th} percentile of the log returns of the S&P500 is located. For an intuitive understanding of our approach, note that the 98^{th} percentile of the distribution considered is included in the CI obtained for the estimate of the VaR at 96%.

4.3 Distorted distributions

To assess an exposure, the first point to consider is to fit an appropriate distribution on the data. Indeed, looking at Figure 10, we observe that the "natural" distribution fitted on the underlying market data set[22] is multi-modal. In this section, we use distributions defined on an infinite support $(-\infty, +\infty)$, and the losses are represented by negative values. In practice, we often observe this kind of pattern for financial or economic data sets. We observe from this graph

[21] For more details, we suggest the previous cited articles

[22] Daily returns Hang Seng Index from 24/07/2006 to 24/07/2008

that we can separate large losses from the other ones and then obtain a better understanding of the probability of these outcomes. In this section we propose an alternative to our proposal in paragraph 3 for the fit of the distributions characterising the risk factors, and by doing so we introduce an alternative risk measure approach.

Given a risk factor X characterised by a distribution function F_X, fitting a multi-modal distributions can be done in several manners. The most natural way of representing multi-modal data is to implement a kernel density strategy, however, if we are interested in capturing other aspects of exposure not captured by data, another possibility is to transform a uni-modal distribution fitted to the data into another one, as illustrated in Figure 10.

Multi-modal distribution construction has been investigated by many statisticians considering mainly multimodal distributions inside the exponential family (Fisher (1922)) and more recently by economists within the dual theory of choice (Yaari (1987)). Both approaches extend the notion of multimodality appearing as a mixture of normal or possibly other unimodal densities and suggest transforming the original distribution into a new one using a distortion function $g(.)$ with appropriate properties. A function $g : [0, 1] \to [0; 1]$ is a distortion function if (i) $g(0) = 0$ and $g(1) = 1$, (ii) g is a continuous increasing function. Different distortion functions have been proposed in the literature. A wide range of parametric families of distortion functions are mentioned in Wang (2000) or Hardy and Wirch (2001). Cobb et al. (1987) also proposes an interesting approach which is more general and whose applicability is based on robust statistical techniques.

We introduce some functions g resulting in a bimodal distribution[23]. To create multi-modality we need to use a function g which possesses saddle points. The saddle point generates a second mode in the new distribution which allows us to take into account different patterns located in the tails. The distortion function g fulfilling this objective can be an inverse S-shaped polynomial function of degree 3 - for instance given by the following equation and characterised by two parameters δ and β:

[23]Here our approach is mainly descriptive, in another paper we provide robust estimation from original data sets using maximum likelihood and the weighted moment method.

$$g_\delta(x) = a\left[\frac{x^3}{6} - \frac{\delta}{2}x^2 + \left(\frac{\delta^2}{2} + \beta\right)x\right]. \tag{4.6}$$

We note that $g_\delta(0) = 0$, and to get $g_\delta(1) = 1$ this implies that the coefficient of normalisation is equal to $a = \left(\frac{1}{6} - \frac{\delta}{2} + \frac{\delta^2}{2} + \beta\right)^{-1}$. The function g_δ will increase if $g'_\delta > 0$ requiring $0 < \delta < 1$. The parameter $\delta \in [0,1]$ allows us to locate the saddle point. The curve exhibits a concave part and a convex part. The parameter $\beta \in \mathbb{R}$ controls the information under each mode in the distorted distribution. Illustration of the role of δ on the location of the saddle points and of β for the shape of this bimodal distribution can be found in Guégan and Hassani (2015*a*) (other functions can be found also in this article). We provide a graph (Figures 11) below which shows the creation of a bi-modal distribution using the transformation $g(F_X)$.

As soon as we have obtained a bi-modal (or multi-modal) distribution, it is interesting to compute a risk using this distribution. In Guégan and Hassani (2015*a*) we suggest to consider the following new risk measure $\rho_g(X)$ (associated with a risk factor X admitting a cumulative distribution $S_X(x) = \mathbb{P}(X > x)$, transformed by a distortion function g), defined by

$$\rho_g(X) = \int_{-\infty}^{0} [g(S_X(x)) - 1]dx + \int_{0}^{+\infty} g(S_X(x))dx, \tag{4.7}$$

as soon as the two integrals are finite. Assuming g is differentiable on $[0,1]$ and $F_X(x)$ is continuous, then a new risk measure - the distortion risk measure - can be re-written as:

$$\rho_g(X) = \mathbb{E}[Xg'(S_X(x))] = \int_0^1 F_X^{-1}(1-p)dg(p) = \mathbb{E}_g[F_X^{-1}]. \tag{4.8}$$

Distortion functions arose from empirical observations that people do not evaluate risk as a linear function of the actual probabilities for different outcomes but rather as a non-linear distortion function. It is used to transform the probabilities of the loss distribution to another probability distribution by re-weighting the original distribution. This transformation increases the weight given to desirable events and deflates others. Different distortions g have been proposed in the literature. A wide range of parametric families of distortion functions have been mentioned in Fisher (1922), and Wang (2000). We can also use specific distortion functions if we want to increase the influence of asymmetry in the transformation of the original distribution and work for instance as follows (Sereda et al. (2010)):

$$\rho_{g_i}(X) = \int_{-\infty}^{0} [g_1(S_X(x)) - 1]dx + \int_{0}^{+\infty} g_2(S_X(x))dx, \quad (4.9)$$

with $g_i(u) = u + k_i(u - u^2)$ for $k \in]0, 1]$ et $\forall i \in \{1, 2\}$. With this approach one models loss and gains differently, relative to the values of the parameters $k_i, i = 1, 2$. Thus upside and downside risks are modelled in different ways. Nevertheless the calibration of the parameters $k_i, i = 1, 2$ remains an open problem. Estimation procedures are provided in a companion paper.

With this approach we obtain another way to use the concept of spectrum risk measure discussed previously. Here the weights are g_1 and g_2: they permit to separate the risks associated for instance to the center from the risks associated to the tail. Thus the corresponding coherent risk measure will be defined as:

$$\rho(X) = \mathbb{E}_g[F_X^{-1}(x)|F_X^{-1}(x) > F_X^{-1}(\delta)]. \quad (4.10)$$

While Figure 10 illustrates the distortion, we provide the quantiles computed on a uni-modal distribution and the VaR computed on the multi-modal distribution. For instance, the empirical 95^{th} percentile equals 0.0316, the Gaussian 95^{th} percentile equals 0.0272 and the distorted 95^{th} percentile equals 0.0433. However, here we are interested in the negative part of the distribution for risk management purposes, and the results are for closer, indeed, the empirical 5^{th} percentile equals -0.0276, the Gaussian 5^{th} percentile equals -0.0286 and the distorted 5^{th} percentile equals -0.027.

The distortion of the distribution induces a mass transfer from one part of the distribution to another, here as we capture the hump in the right tail, the body and the left tail are mechanically impacted. As a result, though the 95^{th} percentile of the distorted distribution is much higher than the others, the risk measure applied to the left tail (the negative part of the distribution) is the lowest. Consequently, we observe that with a better capture of the exposure the risk measure decreases and therefore the capital requirement impact might be non negligible for financial institutions.

5 Conclusion and Recommendations

In the introduction, which analysed several guidelines issued by the EBA and the Basel Committee, we pointed out the fact that the regulators impose specific distributions, risk measures and confidence levels to analyse the risk factors in order to evaluate capital requirements of financial institutions. It appears that their approach is non holistic and their analysis of the risks relies on a disconnection between the components outlined in the previous sentence, i.e. the tools necessary to assess the risks.

In this paper we show that risk measurement in financial institutions depends intrinsically on how the tools are chosen, i.e. the distribution, the combinations of these distributions, the type of risk measure and the level of confidence. Therefore, the existence of a risk measure as discussed in the regulation is questionable, as for example modifying the level of confidence by a few percent would result in completely different interpretations. The regulators fail to propose an appropriate approach to measure these risks in financial institutions as soon as they do not take into account the problem of risk modelling in its globality.

Regulators are far too prescriptive and their choices questionable:

- Imposing distributions does not really make sense whatever the risks to be modelled as these may change quite quickly. We may wonder where these *a priori* are coming from.
- The regulation reflects some misunderstanding regarding distribution properties (probabilistic approach) and of the particular properties surrounding their fittings (statistical approach).
- The levels of confidence p seem rather arbitrary. They neither take into account the flexibility of risk measures nor the impact of the underlying distribution, misleading risk managers.

While these fundamental problems are not addressed, others are completely ignored such as the concept of spectral analysis, spectrum or distortion risk measures (Sereda et al. (2010), Guégan and Hassani (2015*a*)). Despite the cosmetic changes included in Basel II and III, the propositions do not enable a better risk management, and the response of banks to regulatory points is not appropriate as they do not correspond to the reality. It is therefore not surprising that capital calculations and stress testing are still unclear, and that these are not able to capture asymmetric shocks corresponding to extreme incidents.

Some other questions should also be addressed:

- Is it more efficient in terms of risk management to measure the risk and then build a capital buffer or to adjust the risk taken, considering the capital we have? In other words, maybe banks should start optimising their income generation with respect to the capital they already have.

- The previous points are all based on uni-modal parametric distributions to characterise each risk factor. What is the impact of using multi-modal distributions in terms of risk measurement and management? We believe that an empirical evaluation of the risks provides bank with a reliable benchmark and a starting point in terms of what would be an acceptable capital charge or risk assessment.

- One of the biggest issues lies in the fact that we do not know how to combine or aggregate $VaR(X)_{p_1}$, $VaR(Y)_{p_2}$ and $VaR(Z)_{p_3}$ evaluated on three different kinds of risks at three different confidence levels p_1, p_2, p_3. This mechanically prevents banks from building a holistic approach from a capital point of view. How should we proceed to solve the problem, should we use $p = max(p_1, p_2, p_3)$, or the minimum or the average?

- Although in this paper we have focused on each factor taken independently, the question of dependence is quite important too. Maybe not as important as the impact of the distribution selected for the risk factor (Guégan and Hassani (2013)) but not addressing this issue properly could lead to a mis-interpretation of the results. The choice of the copula has a direct impact on the dependence structure we would like to apply and the capture of shocks. For instance, a Gaussian or Student t-copula is symmetric, despite the fact that a t-copula with a low number of degrees of freedom could capture tail dependencies; these would not capture asymmetric shocks. Archimedean or extreme value copulas associated with a vine strategy would be more appropriate (Guégan and Maugis (2010*b*)).

- In a situation such as one depicted by the stress-testing process with a forward looking perspective, if the risks are not correctly measured then the foundations will be very fragile and the outcome of the exercise not reliable. Indeed, stressing a situation requires an appropriate initial assessment of the real exposure, otherwise the stress would merely model what should have been captured originally and therefore be useless (Bensoussan et al. (2015), Guégan and Hassani (2015*b*), Hassani (2015)).

We came up to the conclusion that the debate related to the selection of a risk measure over another is not really relevant, and considering issues raised in the previous sections our main recommendation would be to leave as much flexibility as possible to the modellers to build the most appropriate models for risk management purposes initially and then extend with conservative buffers for capital purposes. The objective would be to suggest that good risk management would mechanically limit the exposures and the losses and therefore ultimately reduce the regulatory capital burden. Models should only be a reflection of the underlying risk framework and not a tool to justify a reduced capital charge. We would like to see the supervisory face of the authorities more and their regulatory face less; in other words we would like them to stop focusing so much on a bank's risk measurement comparability and more on financial institutions risk understanding. It would probably be wise if both regulators and risk managers worked together (e.g., academic formation open to both corpus, regular workshops, etc., (Guégan (2009))) rather than as opponents, in order to reach their objective of stability of the financial system first and profitability second.

Finally, we believe that the implementation of combinations of risk measures such as the spectral risk measures, spatial VaR, risk measure spectrum or the distortion of the risk measure may help addressing the limitations, the inefficiencies or blind spots of the more traditional risk measures for instance the VaR or the ES. Indeed, the combinations help capturing a more diffuse risk, and not a specific value in a spot, providing a better representation of the exposure, incorporating the uncertainty related to the selection of the distribution used to assess the risks and the fittings. Furthermore, they allow capturing the multi-modality of some distributions. Besides, the combination also smooths the risk measurement reducing the volatility of these over time. Consequently, we would suggest financial institutions to start implementing the methodology to measure their risk more accurately, and regulators to start considering them for regulatory capital calculations. As presented in the previous section, it is really important to understand that capturing the exposure more accurately does not necessarily lead to larger regulatory capital, but mechanically to better risk management.

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Parameters \ Distribution	LogNormal	Weibull	GPD	GH	α-Stable	GEV	GEVbm
μ	4.412992	-	1541.558	-5.5846599	-	587.855749	50.272385
σ	1.653022	-	-	-	-	2137.940297	64.720971
α	-	0.5895611	-	0.1906536	0.86700	-	-
β	-	182.9008432	1185.8083087	0.1906304	0.95000	-	-
ξ	-	-	0.9039617	-	675 923	3.634536	1.030459
δ	-	-	-	22.5118547	58.18019	-	-
λ	-	-	-	-0.871847		-	-
γ	-	-	-	-	54.21489	-	-
KS	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16	< 2.2e-16
AD	6.117e-09	6.117e-09	6.117e-09	NA	6.117e-09 (d)	6.117e-09	6.117e-09

Table 1 – Parameter estimates

This table provides the estimated parameters for the seven parametric distributions fitted on the operational risk data set. If for the α stable distribution $\alpha < 1$, for the GEV, GEVbm and GPD $\xi > 1$ we are in the presence of an infinite mean model. The p-values of both Kolmogorov-Smirnov and Anderson-Darling tests are also provided for the fit of each distribution.

X_1 \ X_2		LogNormal		Weibull		GPD		GH		Alpha-Stable		GEV		GEVbm	
		VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}
LogNormal	p_1	2 493	6 103	2 328	4 820	75 546	217 916	2 618	7 753	2 552	44 407	2.834332e+07	1.721696e+21	2 658	19 225
	p_2	4 087	9 057	3 528	6 805	93 442	353 270	4 662	12 064	4 731	85 397	3.665483e+08	3.443392e+21	4 872	34 914
	p_3	7 252	14 693	5 735	10 419	142 038	716 560	9 291	20 539	10 725	203 193	1.072611e+10	8.608479e+21	10 723	76 829
	p_4	24 250	43 132	16 710	27 468	746 560	4 871 970	36 925	59 361	111 991	1 789 756	5.390204e+13	8.608479e+22	83 960	551 875
Weibull	p_1	-	-	2 230	3 637	75 270	219 140	2 424	6 507	2 345	80 360	2.874992e+07	1.648438e+21	2 459	32 126
	p_2	-	-	3 110	4 663	92 818	356 117	4 019	9 939	4 029	157 714	3.746901e+08	3.296876e+21	4 202	61 102
	p_3	-	-	4 425	6 176	140 307	725 572	7 651	16 774	9 003	385 667	1.062522e+10	8.242191e+21	8 856	143 962
	p_4	-	-	8 542	10 777	744 266	4 982 719	29 929	48 745	102 987	3 644 010	4.051415e+13	8.242191e+22	81 979	1 249 637
GPD	p_1	-	-	-	-	150 276	481 590	75 882	198 596	76 003	253 293	2.900050e+07	8.175899e+21	2.852749e+07	8.991075e+20
	p_2	-	-	-	-	185 210	799 161	93 953	314 173	94 929	423 142	3.804500e+08	1.635180e+22	3.719388e+08	1.798215e+21
	p_3	-	-	-	-	281 725	1 667 960	143 016	618 297	149 135	886523	1.055955e+10	4.087950e+22	1.038451e+10	4.495538e+21
	p_4	-	-	-	-	1 482 342	12 136 572	735 168	3 911 547	869 960	6 369 873	5.299107e+13	4.087950e+23	4.773943e+13	4.495538e+22
GH	p_1	-	-	-	-	-	-	2 784	9 385	2 705	73 357	2.880134e+07	3.149731e+20	2.875817e+07	6.426626e+19
	p_2	-	-	-	-	-	-	5 338	14 981	5 470	142 939	3.771517e+08	6.299462e+20	3.674012e+08	1.285325e+20
	p_3	-	-	-	-	-	-	11 469	25 984	13 280	344 980	1.092553e+10	1.574866e+21	1.088133e+10	3.213313e+20
	p_4	-	-	-	-	-	-	47 282	75 037	120 486	3 167 054	5.156340e+13	1.574866e+22	4.991674e+13	3.213313e+21
Alpha-Stable	p_1	-	-	-	-	-	-	-	-	2 649	146 535	2.868921e+07	2.877576e+22	2 822	31 932
	p_2	-	-	-	-	-	-	-	-	5 648	289 288	3.667449e+08	5.755152e+22	5 644	59 930
	p_3	-	-	-	-	-	-	-	-	15 890	709 490	1.007457e+10	1.438788e+23	13 339	137 177
	p_4	-	-	-	-	-	-	-	-	225 543	6 659 012	3.907140e+13	1.438788e+24	96 356	1 107 042
GEV	p_1	-	-	-	-	-	-	-	-	-	-	1.822300e+08	1.096423e+28	2.875582e+07	1.635820e+19
	p_2	-	-	-	-	-	-	-	-	-	-	2.615063e+09	2.192846e+28	3.640431e+08	3.271639e+19
	p_3	-	-	-	-	-	-	-	-	-	-	8.028848e+10	5.482116e+28	1.075443e+10	8.179098e+19
	p_4	-	-	-	-	-	-	-	-	-	-	4.437624e+14	5.482116e+29	4.820065e+13	8.179098e+20
GEVbm	p_1	-	-	-	-	-	-	-	-	-	-	-	-	2 894	59 150
	p_2	-	-	-	-	-	-	-	-	-	-	-	-	5 985	114 221
	p_3	-	-	-	-	-	-	-	-	-	-	-	-	15 597	271 596
	p_4	-	-	-	-	-	-	-	-	-	-	-	-	170 339	2 336 019

Table 2 – Correlated Risk Measures

This table presents the VaRs and the ESs obtained on fully correlated random variables simulated with seven distributions.

LN-LN 1	393	663	$1,373$	$2,503$	$7,721$	$11,661$	$27,292$
LN-LN 2	395	667	$1,376$	$2,503$	$7,721$	$11,677$	$27,517$
LN-WE 1	447	742	$1,439$	$2,427$	$6,299$	$8,924$	$18,498$
LN-WE 2	564	826	$1,374$	$2,068$	$4,654$	$6,406$	$14,066$
LN-GPD 1	$4,321$	$6,181$	$11,432$	$21,158$	$88,382$	$163,788$	$689,569$
LN-GPD 2	$58,968$	$60,766$	$65,759$	$74,945$	$138,510$	$209,859$	$726,643$
LN-GH 1	364	611	$1,313$	$2,569$	$9,882$	$16,037$	$41,329$
LN-GH 2	480	742	$1,418$	$2,528$	$8,205$	$12,765$	$30,592$
LN-AS 1	377	614	$1,269$	$2,461$	$10,965$	$21,402$	$111,987$
LN-AS 2	476	725	$1,374$	$2,472$	$9,657$	$18,319$	$101,929$
LN-GEV 1	$25,132$	$137,464$	$2,097,977$	$28,700,959$	$10.73e9$	$134.51e9$	$47,029e9$
LN-GEV 2	$25,313$	$138,221$	$2,095,098$	$29,156,891$	$10.47e9$	$135.38e9$	$45,501e9$
LN-GEVbm 1	366	614	$1,312$	$2,579$	$11,037$	$20,542$	$91,109$
LN-GEVbm 2	481	742	$1,423$	$2,571$	$9,670$	$17,603$	$80,694$

Table 3 – VaR(X) + VaR(Y) vs VaR(X + Y) part 1

The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

WE-WE 1	501	820	$1,505$	$2,352$	$4,878$	$6,187$	$9,703$
WE-WE 2	501	821	$1,510$	$2,352$	$4,879$	$6,185$	$9,807$
WE-GPD 1	$4,376$	$6,259$	$11,498$	$21,082$	$86,961$	$161,051$	$680,774$
WE-GPD 2	$58,916$	$60,639$	$65,520$	$74,662$	$138,368$	$209,701$	$726,035$
WE-GH 1	418	690	$1,379$	$2,494$	$8,460$	$13,300$	$32,534$
WE-GH 2	533	795	$1,379$	$2,208$	$6,472$	$10,534$	$27,998$
WE-AS 1	431	692	$1,335$	$2,386$	$9,544$	$18,665$	$103,193$
WE-AS 2	528	779	$1,341$	$2,148$	$7,556$	$16,025$	$101,095$
WE-GEV 1	$25,186$	$137,542$	$2,098,044$	$28,700,884$	$10.73e9$	$134.51e9$	$47,029e9$
WE-GEV 2	$25,197$	$138,107$	$2,094,946$	$29,156,852$	$10.47e9$	$135.38e9$	$45,501e9$
WE-GEVbm 1	420	692	$1,379$	$2,504$	$9,616$	$17,805$	$82,315$
WE-GEVbm 2	534	796	$1,381$	$2,237$	$7,710$	$15,281$	$79,250$

Table 4 – VaR(X) + VaR(Y) vs VaR(X + Y) part 2

The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

GPD-GPD 1	8,250	11,699	21,490	39,812	169,044	315,915	1,351,846
GPD-GPD 2	117,080	120,546	130,394	148,749	276,271	418,831	1,452,006
GPD-GH 1	4,292	6,129	11,372	21,224	90,543	168,164	703,606
GPD-GH 2	59,005	60,888	66,096	75,538	139,002	209,869	726,103
GPD-AS 1	4,305	6,131	11,328	21,116	91,627	173,528	774,264
GPD-AS 2	58,987	60,890	66,273	76,314	147,644	229,984	834,971
GPD-GEV 1	29,061	142,981	2,108,036	28,719,614	$10.73e9$	$134.51e9$	$47,029e9$
GPD-GEV 2	92,215	210,767	2,181,852	29,254,626	$10.47e9$	$135.38e9$	$45,501e9$
GPD-GEVbm 1	4,292	6,129	11,372	21,224	90,543	168,164	703,606
GPD-GEVbm 2	59,005	60,888	66,096	75,538	139,002	209,869	726,103
GH-GH 1	335	559	1,253	2,635	12,043	20,413	55,366
GH-GH 2	335	559	1,253	2,635	12,043	20,413	55,366
GH-AS 1	348	562	1,209	2,527	13,126	25,778	126,024
GH-AS 2	442	683	1,393	2,778	12,596	23,446	104,497
GH-GEV 1	25,103	137,412	2,097,918	28,701,025	$10.73e9$	$134.51e9$	$47,029e9$
GH-GEV 2	25,635	138,429	2,095,206	29,157,735	$10.47e9$	$135.38e9$	$45,501e9$
GH-GEVbm 1	336	562	1,252	2,645	13,198	24,917	105,146
GH-GEVbm 2	446	703	1,451	2,895	12,502	22,224	84,680

Table 5 – VaR(X) + VaR(Y) vs VaR(X + Y) part 3

The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

AS-AS 1	361	564	$1,165$	$2,419$	$14,210$	$31,142$	$196,682$
AS-AS 2	360	562	$1,159$	$2,428$	$14,153$	$31,459$	$201,447$
AS-GEV 1	$25,116$	$137,414$	$2,097,873$	$28,700,918$	$10.73e9$	$134.51e9$	$47,029e9$
AS-GEV 2	$26,139$	$140,091$	$2,099,977$	$29,175,188$	$10.47e9$	$135.38e9$	$45,501e9$
AS-GEVbm 1	349	564	$1,208$	$2,537$	$14,282$	$30,282$	$175,804$
AS-GEVbm 2	443	683	$1,399$	$2,849$	$15,645$	$33,285$	$189,589$
GEV-GEV 1	$49,871$	$274,264$	$4,194,582$	$57,399,416$	$21.46e9$	$269e9$	$94,058e9$
GEV-GEV 2	$49,844$	$275,821$	$4,189,583$	$58,313,419$	$20.94e9$	$271e9$	$91,002e9$
GEV-GEVbm 1	$25,105$	$137,414$	$2,097,917$	$28,701,036$	$10.73e9$	$134.51e9$	$47,029e9$
GEV-GEVbm 2	$26,105$	$139,855$	$2,099,195$	$29,174,309$	$10.47e9$	$135.38e9$	$45,501e9$
GEVbm-GEVbm 1	338	564	$1,252$	$2,656$	$14,353$	$29,422$	$154,927$
GEVbm-GEVbm 2	340	565	$1,251$	$2,663$	$14,609$	$29,967$	$158,273$

Table 6 – VaR(X) + VaR(Y) vs VaR(X + Y) part 4

The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

LN-LN 1	$1,895$	$2,587$	$4,226$	$6,616$	$16,572$	$23,652$	$50,725$
LN-LN 2	$1,895$	$2,587$	$4,226$	$6,616$	$16,572$	$23,652$	$50,725$
LN-WE 1	$1,727$	$2,302$	$3,574$	$5,293$	$11,739$	$16,021$	$31,500$
LN-WE 2	$1,541$	$1,970$	$2,882$	$4,092$	$8,841$	$12,269$	$25,675$
LN-GPD 1	$87,496$	$101,478$	$140,329$	$211,059$	$682,080$	$1,191,608$	$4,513,150$
LN-GPD 2	$87,065$	$100,726$	$138,767$	$208,277$	$674,213$	$1,180,157$	$4,488,157$
LN-GH 1	$2,146$	$2,984$	$5,081$	$8,347$	$23,114$	$33,777$	$71,406$
LN-GH 2	$1,996$	$2,698$	$4,383$	$6,898$	$17,681$	$25,214$	$50,397$
LN-AS 1	$16,694$	$24,801$	$48,732$	$95,726$	$459,981$	$905,044$	$4,350,967$
LN-AS 2	$16,545$	$24,525$	$48,067$	$94,322$	$454,147$	$895,398$	$4,326,497$
LN-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
LN-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
LN-GEVbm 1	$5,608$	$8,174$	$15,460$	$29,105$	$126,073$	$237,314$	$1,032,332$
LN-GEVbm 2	$5,457$	$7,888$	$14,762$	$27,640$	$120,229$	$227,765$	$1,008,148$

Table 7 – ES(X) + ES(Y) vs ES(X + Y) part 1

The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

WE-WE 1	$1,559$	$2,016$	$2,921$	$3,970$	$6,905$	$8,390$	$12,276$
WE-WE 2	$1,559$	$2,016$	$2,921$	$3,970$	$6,905$	$8,390$	$12,276$
WE-GPD 1	$87,328$	$101,193$	$139,676$	$209,736$	$677,247$	$1,183,977$	$4,493,926$
WE-GPD 2	$86,887$	$100,505$	$138,515$	$208,044$	$674,087$	$1,180,072$	$4,488,101$
WE-GH 1	$1,978$	$2,698$	$4,428$	$7,024$	$18,280$	$26,146$	$52,182$
WE-GH 2	$1,810$	$2,389$	$3,739$	$5,758$	$15,192$	$22,257$	$46,312$
WE-AS 1	$16,526$	$24,516$	$48,079$	$94,403$	$455,148$	$897,413$	$4,331,742$
WE-AS 2	$16,359$	$24,217$	$47,423$	$93,172$	$452,023$	$893,523$	$4,325,897$
WE-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2447e18$
WE-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2447e18$
WE-GEVbm 1	$5,440$	$7,889$	$14,807$	$27,782$	$121,240$	$229,683$	$1,013,108$
WE-GEVbm 2	$5,270$	$7,579$	$14,119$	$26,506$	$118,106$	$225,770$	$1,007,256$

Table 8 – ES(X) + ES(Y) vs ES(X + Y) part 2

The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

GPD-GPD 1	$173,097$	$200,369$	$276,431$	$415,503$	$1,347,588$	$2,359,564$	$8,975,575$
GPD-GPD 2	$173,097$	$200,369$	$276,431$	$415,503$	$1,347,588$	$2,359,564$	$8,975,575$
GPD-GH 1	$87,747$	$101,874$	$141,183$	$212,791$	$688,622$	$1,201,732$	$4,533,832$
GPD-GH 2	$87,330$	$101,092$	$139,298$	$208,887$	$674,421$	$1,180,208$	$4,488,112$
GPD-AS 1	$102,295$	$123,692$	$184,834$	$300,169$	$1,125,489$	$2,073,000$	$8,813,392$
GPD-AS 2	$101,891$	$122,938$	$182,933$	$295,782$	$1,098,582$	$2,016,042$	$8,499,442$
GPD-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GPD-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GPD-GEVbm 1	$91,209$	$107,065$	$151,562$	$233,548$	$791,581$	$1,405,270$	$5,494,758$
GPD-GEVbm 2	$90,787$	$106,267$	$149,558$	$229,042$	$766,781$	$1,355,085$	$5,243,081$
GH-GH 1	$2,397$	$3,380$	$5,935$	$10,078$	$29,655$	$43,901$	$92,088$
GH-GH 2	$2,397$	$3,380$	$5,935$	$10,078$	$29,655$	$43,901$	$92,088$
GH-AS 1	$16,945$	$25,197$	$49,586$	$97,457$	$466,523$	$915,168$	$4,371,648$
GH-AS 2	$16,809$	$24,941$	$48,924$	$95,926$	$458,199$	$899,741$	$4,327,096$
GH-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GH-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GH-GEVbm 1	$5,858$	$8,571$	$16,314$	$30,836$	$132,615$	$247,439$	$1,053,014$
GH-GEVbm 2	$5,722$	$8,305$	$15,616$	$29,227$	$124,294$	$232,340$	$1,009,422$

Table 9 – ES(X) + ES(Y) vs ES(X + Y) part 3

The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

AS-AS 1	$31,493$	$47,015$	$93,237$	$184,836$	$903,390$	$1,786,436$	$8,651,209$
AS-AS 2	$31,493$	$47,015$	$93,237$	$184,836$	$903,390$	$1,786,436$	$8,651,209$
AS-GEV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
AS-GEV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
AS-GEVbm 1	$20,406$	$30,388$	$59,965$	$118,215$	$569,482$	$1,118,706$	$5,332,574$
AS-GEVbm 2	$20,270$	$30,130$	$59,302$	$116,655$	$559,704$	$1,097,691$	$5,212,237$
GEV-GEV 1	$16e18$	$24e18$	$48e18$	$97e18$	$489e18$	$979e18$	$4,895e18$
GEV-GEV 2	$16e18$	$24e18$	$48e18$	$97e18$	$489e18$	$979e18$	$4,895e18$
GEV-GEVbm 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GEV-GEVbm 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GEVbm-GEVbm 1	$9,320$	$13,761$	$26,693$	$51,593$	$235,574$	$450,977$	$2,013,940$
GEVbm-GEVbm 2	$9,320$	$13,761$	$26,693$	$51,593$	$235,574$	$450,977$	$2,013,940$

Table 10 – ES(X) + ES(Y) vs ES(X + Y) part 4

The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couples of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GEV = Generalised Extreme Value, GEVbm = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. We use the distributions fitted on the data set representing "Execution, Delivery, and process Management".

Distribution / %tile	Empirical		LogNormal		Weibull		GPD		GH		Alpha-Stable		GEV		GEVbm	
	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES
90%	575	2 920	686	2 090	753	1 455	10 745	146 808	627	2 950	582	29 847 $+\infty$	2.097291e+06	$+\infty$	626	12 755
95%	1 068	5 051	1 251	3 264	1 176	1 979	19 906	224 872	1 317	5 006	1 209	58 872 $+\infty$	2.869971e+07	$+\infty$	1 328	24 616
97.5%	1 817	8 775	2 106	4 925	1 674	2 572	37 048	368 703	2 608	8 187	2 563	116 016 $+\infty$	3.735524e+08	$+\infty$	2 762	47 356
99%	3 662	18 250	3 860	8 148	2 439	3 468	84 522	758 667	5 917	14 721	7 105	283 855 $+\infty$	1.073230e+10	$+\infty$	7 177	111 937
99.9%	31 560	104 423	13 646	24 784	4 852	6 191	675 923	5 328 594	28 064	46 342	98 341	2 649 344 $+\infty$	4.702884e+13	$+\infty$	77 463	945 720

Table 11 – Univariate Risk Measures

This table exhibits the VaRs and ESs for the height types of distributions considered - for instance empirical, lognormal, Weibull, GPD, GH, α-stable, GEV and GEV fitted on a series of maxima - for five confidence level (for instance, 90%, 95%, 97.5%, 99% and 99.9%) evaluated on the period 2009-2014. Note that the parameters obtained for the α-stable and the GEV fitted on the entire data set lead to infinite mean model and therefore, the ES are hardly applicable.

80.00%	81.00%	82.00%	83.00%	84.00%	85.00%	86.00%	87.00%
-81.843	-74.943	-66.539	-57.410	-47.461	-35.212	-20.496	-3.984
88.00%	89.00%	90.00%	91.00%	92.00%	93.00%	94.00%	95.00%
16.247	40.129	67.997	102.443	144.756	196.882	266.676	360.135
96.00%	97.00%	98.00%	99.00%	99.50%	99.90%	99.95%	99.99%
489.356	677.618	1,011.196	1,696.400	2,581.672	4,858.396	5,761.766	10,964.930

Table 12 – VaR(X) + VaR(Y) vs VaR(X + Y) valuation - Weibull and lognormal

This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X has been generated using a Weibull and Y has been obtained from a lognormal distribution. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold.

80.00%	81.00%	82.00%	83.00%	84.00%	85.00%	86.00%	87.00%
-86.104	-82.891	-80.004	-75.764	-69.887	-63.385	-55.082	-45.380
88.00%	89.00%	90.00%	**91.00%**	92.00%	93.00%	94.00%	95.00%
-34.810	-21.030	-2.510	**23.340**	54.970	99.660	159.200	249.830
96.00%	97.00%	98.00%	99.00%	99.50%	99.90%	99.95%	**99.99%**
393.730	632.630	1,098.500	2,170.800	3,052.900	4,784.190	17,905.440	**−633,422.500**

Table 13 – VaR(X) + VaR(Y) vs VaR(X + Y) valuation - Weibull & Alpha

This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X has been generated using a Weibull and Y has been obtained from an Alpha-stable distribution. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold.

-0.012	0.022	-0.013	-0.018	-0.015	-0.031	-0.020	-0.026
-0.038	**0.011**	0.028	0.023	0.022	0.024	0.044	0.073
0.074	0.080	0.139	0.144	0.194	0.171	0.167	0.163
0.142	0.141	0.134	0.150	0.179	0.175	0.105	0.107
0.016	**-0.001**	-0.002	-0.003	0.013	-0.021	-0.048	-0.011
-0.016	**0.045**	0.074	0.032	0.074	0.166	0.124	0.104
0.098	0.019	**-0.037**	-0.079	-0.100	-0.120	-0.144	-0.047
-0.070	-0.086	-0.136	-0.234	-0.291	-0.352	-0.272	-0.197
-0.098	0.038	0.121	-0.313	-0.299	-0.483	-0.621	-0.422
-0.457	**0.099**	0.272	0.381	0.430	0.656	0.754	0.533
0.693	1.035	0.715	1.087	0.778	$\mathbf{-0.167}$	-0.479	-0.522
-0.759	-3.391	-2.265	-4.190	-3.137	-6.484	-1.975	**9.502**
6.873	16.636	69.495	50.091	$7,118.689$	$8,798.144$	$\mathbf{-148,979.500}$	NA

Table 14 – VaR(X) + VaR(Y) vs VaR(X + Y) valuation - GEV & GEV

This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X and Y have been obtained on 2 identical GEV distributions. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold. The percentiles represented are sequentially going from 1% to 99% by 1%, and to capture the tail, the 99.95th, 99.9th, 99.95th and 99.99th percentiles are added.

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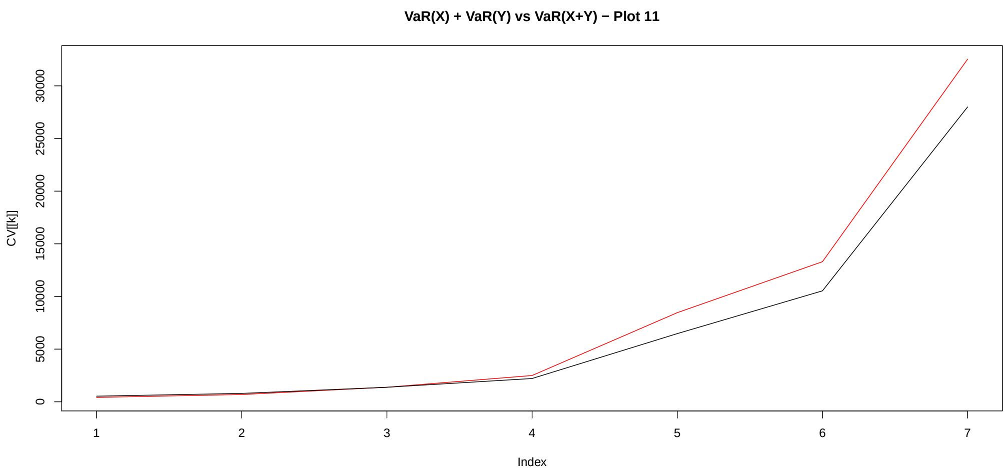

Figure 1 – VaR(X) + VaR(Y) vs VaR(X + Y) Plot - Weibull & GH

This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using a Weibull distribution and Y has been obtained from a Generalised Hyberbolic distribution. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR seems to be sub-additive.

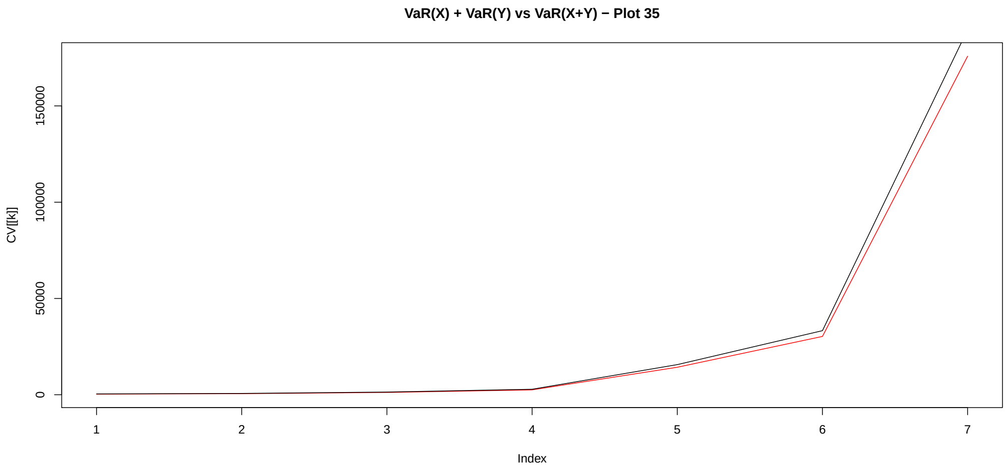

Figure 2 – VaR(X) + VaR(Y) vs VaR(X + Y) Plot - Alpha & GEV max

This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using an Alpha-stable distribution and Y has been obtained from a GEV distribution calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR is not sub-additive.

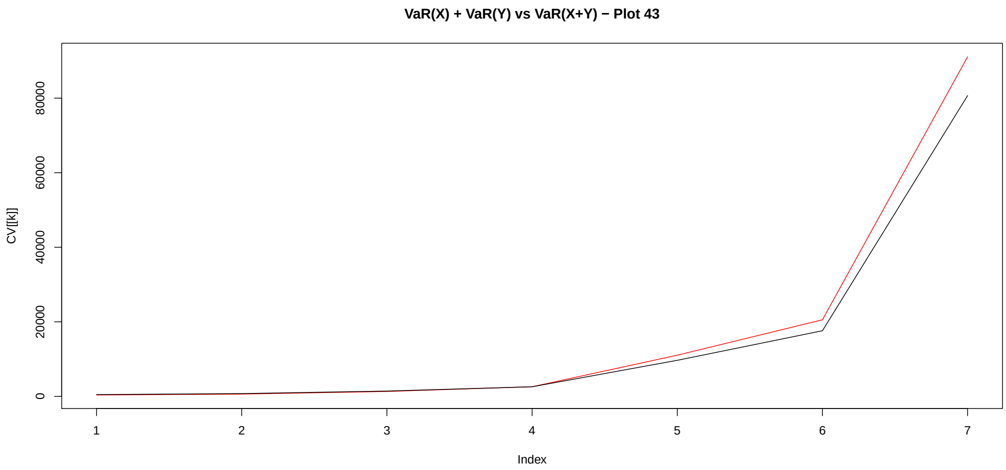

Figure 3 – VaR(X) + VaR(Y) vs VaR(X + Y) Plot - GEV & GEV

This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variables X and Y have been obtained from two identical GEV distributions. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR is sub-additive.

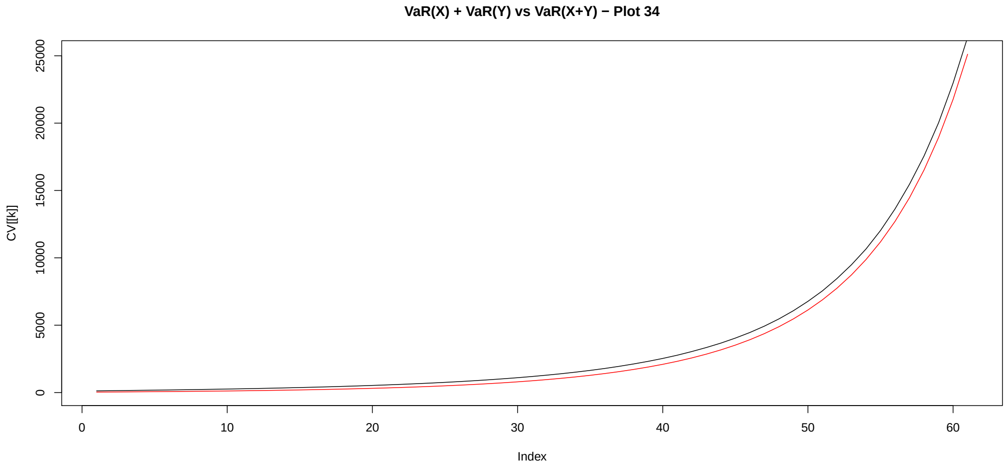

Figure 4 – VaR(X) + VaR(Y) vs VaR(X + Y) Plot - Alpha & GH enlarged

This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using an Alpha-stable distribution and Y has been obtained from a Generalised Hyperbolic distribution. The percentiles represented are sequentially going from the 10th to the 70th with a step of 1% between two points. The VaR represented are never sub-additive.

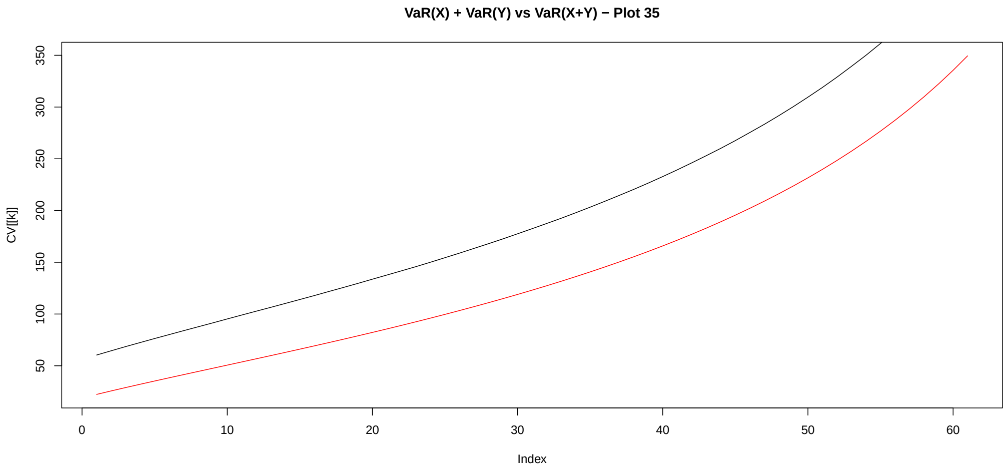

Figure 5 – VaR(X) + VaR(Y) vs VaR(X + Y) Plot - Alpha & GEV max enlarged

This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using a Alpha-stable distribution and Y has been obtained from a GEV distribution calibrated on maxima. The percentiles represented are sequentially going from the 10th to the 70th with a step of 1% between two points. The VaR represented are never sub-additive.

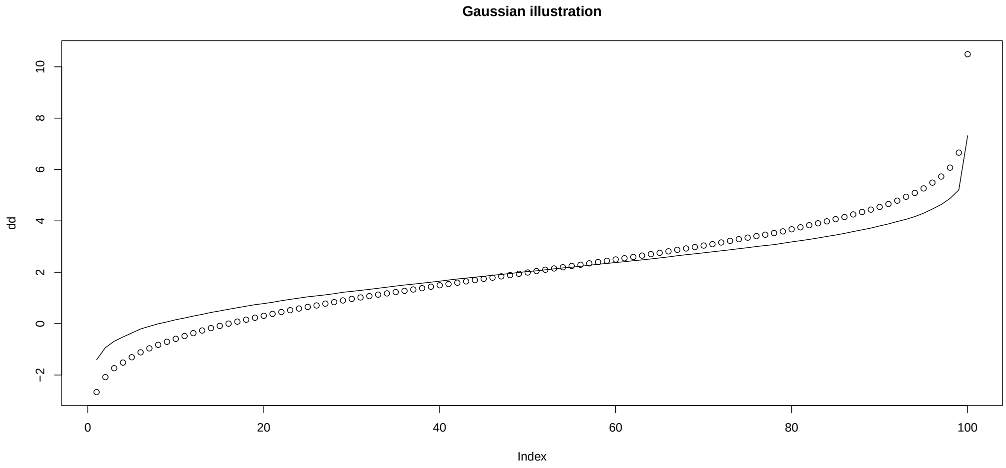

Figure 6 – VaR(X) + VaR(Y) vs VaR(X + Y) Plot - Gaussian case

This plot represents the sum of VaR(X) and VaR(Y) (doted line) versus VaR(X + Y) (solid line). The random variable X has been generated using a Gaussian distribution (0, 1) and Y has been obtained from a Gaussian distribution (2, 1). The VaRs represented are always sub-additive.

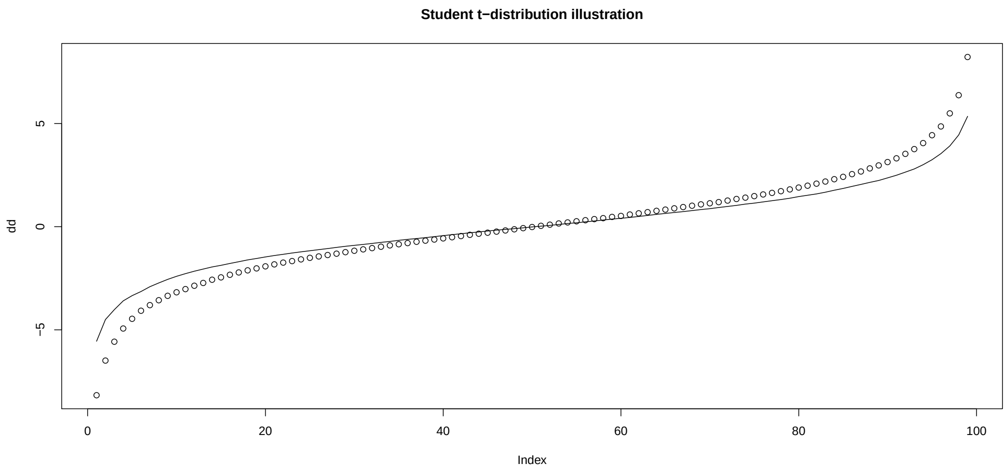

Figure 7 – VaR(X) + VaR(Y) vs VaR(X + Y) Plot - Student case

This plot represents the sum of VaR(X) and VaR(Y) (doted line) versus VaR(X + Y) (solid line). The random variable X has been generated using a Student-t distribution (3 df) and Y has been obtained from a Student-t distribution (4 df). The VaRs represented are always sub-additive.

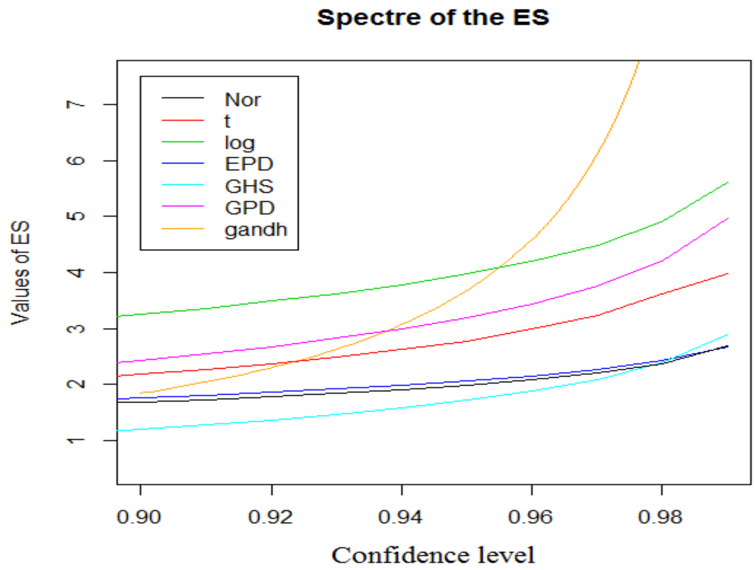

Figure 8 – This figure illustrates how the ES evolves depending on the distribution used.

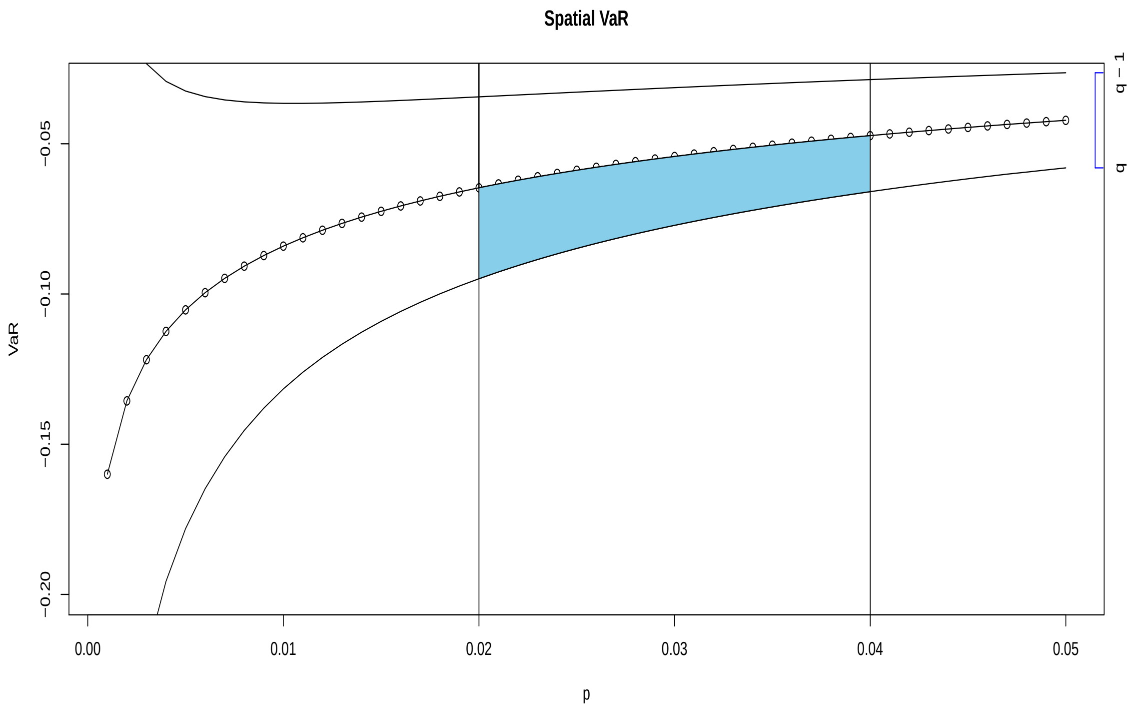

Figure 9 – Spatial VaR

We provide the area corresponding to the SVaR, the lower bound corresponds to VaR_{p_i} and the upper bound to the upper bound of CI_{q_i,p_i}, $i = 1, ..., k$.. The area in blue represent the Spatial VaR capturing simultaneously the uncertainty related to the choice of the distribution and the uncertainty related to the estimation of the parameters. The VaR has been evaluated on the distribution of the return of S&P500 from 01/01/2008 to 31/01/2008.

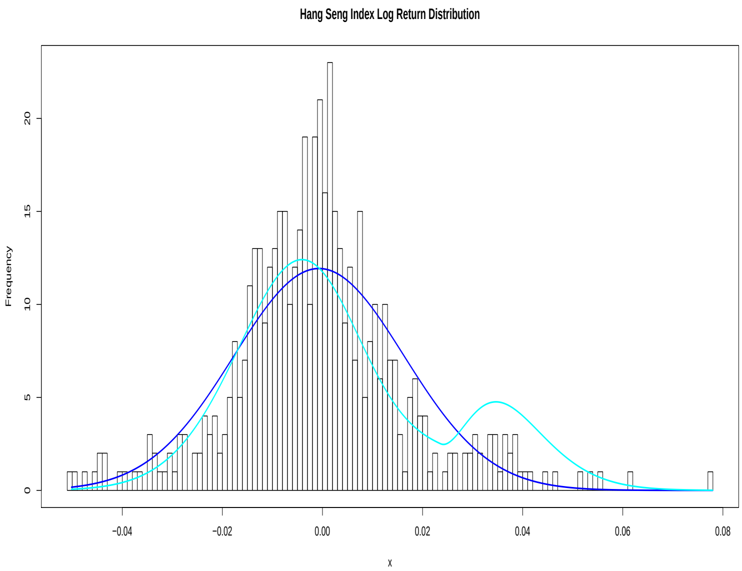

Figure 10 – Hang Seng Index return density

This figure presents the density of the Hang Seng Index. We observe that this one cannot be characterised by a Gaussian distribution, or any distribution that does not capture humps for that matter. Indeed, over the histogram are exhibited the fitted Gaussian distribution in blue and the distorted version in purple allowing to assess the distorted risk measure. This distribution has been obtained using the daily return of the Hang Seng from 24/07/2006 to 24/07/2008.

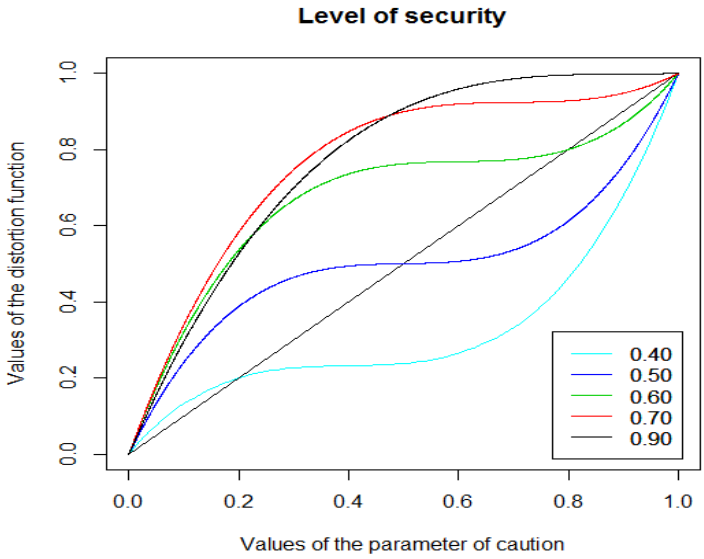

Figure 11 – Distortion function illustration

Curves of the distortion function g_δ introduced in equation (5) for several values of δ and fixed values of $\beta = 0.001$.